

Principles of Macroeconomics: A Production Economy Part 1

Class 4

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► Announcements:

- You should be able to do LC 7 and GH 7 (due September 12th at 11:59pm)
- By the end of today we will be halfway through the material for LC 9 and GH 9 (also due September 12th)

► Topics:

- The production function
- Simple model of production

► Readings:

- Chapter 9.2 (Sources of Long-Run Growth), chapters 9.3-9.4

Why Math?

- ▶ Remember, I said we'd only use math to clarify ideas
- ▶ So why use math here?
 - We want to posit a theory of why countries grow faster than others
 - We could propose all sorts of mechanisms
 - How do we make sure those mechanisms are coherent (as in, will they actually lead to growth given a set of assumptions?)
 - We use math to demonstrate that logic

The Production Function

- ▶ Output: Y
 - Think of this as GDP
- ▶ Production:

$$Y = F(K, L) = AK^{\alpha}L^{1-\alpha}$$

- $K \equiv$ capital
- $L \equiv$ labor hours
- $A \equiv$ total productivity – scales the inputs
- $\alpha \equiv$ how important capital is to production (“capital share of production”)

Production Functions in General

- ▶ $F(K, L)$ can be any function describing how inputs get to outputs
- ▶ The particular function on the previous slide is called a **Cobb-Douglas production function**
- ▶ For example, suppose $Y := \text{cars}$, and $K := \text{factories}$, and $L := \text{worker work days}$. Let $A = 2000$ and $\alpha = 1/3$.
 - What is Y if we have 8 factories and 27 worker work days?

$$\begin{aligned} Y &= 2000 \times 8^{1/3} \times 27^{2/3} \\ &= 2000 \times 2 \times 9 \\ &= 36,000 \text{ cars} \end{aligned}$$

- ▶ What happens if we increase each input by $x\%$?

$$F(2K, 2L) = A(2K)^{1/3}(2L)^{2/3}$$

- ▶ We can simplify this by pulling the 2 out front:

$$\begin{aligned} F(2K, 2L) &= 2AK^{1/3}L^{2/3} \\ &= 2F(K, L) \end{aligned}$$

- ▶ So doubling inputs also doubles the output – **Constant returns to scale**

- ▶ Let $Y = AK^\alpha L^\beta$ – more general Cobb-Douglas. Then:
 - **Decreasing returns:** $\alpha + \beta < 1$
If we double inputs, we get less than double output
 - **Increasing returns:** $\alpha + \beta > 1$
If we double inputs, we get more than double output
 - **Constant returns:** $\alpha + \beta = 1$
If we double inputs, we double output
- ▶ Note that this is for doubling **all** inputs

What if we only change one input?

1. Double K , keep L fixed

$$\begin{aligned}F(2K, L) &= A(2K)^{1/3}L^{2/3} \\&= A2^{1/3}K^{1/3}L^{2/3} \\&< 2AK^{1/3}L^{2/3}\end{aligned}$$

2. Covid Labor Shock – L falls by 16% in one quarter

$$Y_t = AK^{1/3}L^{2/3} \qquad Y_{t+1} = AK^{1/3}(0.84L)^{2/3}$$

Output growth is:

$$\frac{Y_{t+1}}{Y_t} = \frac{AK^{1/3}(0.84L)^{2/3}}{AK^{1/3}L^{2/3}} = 0.89$$

So the model predicts output fell by 11% (reality: 10%)

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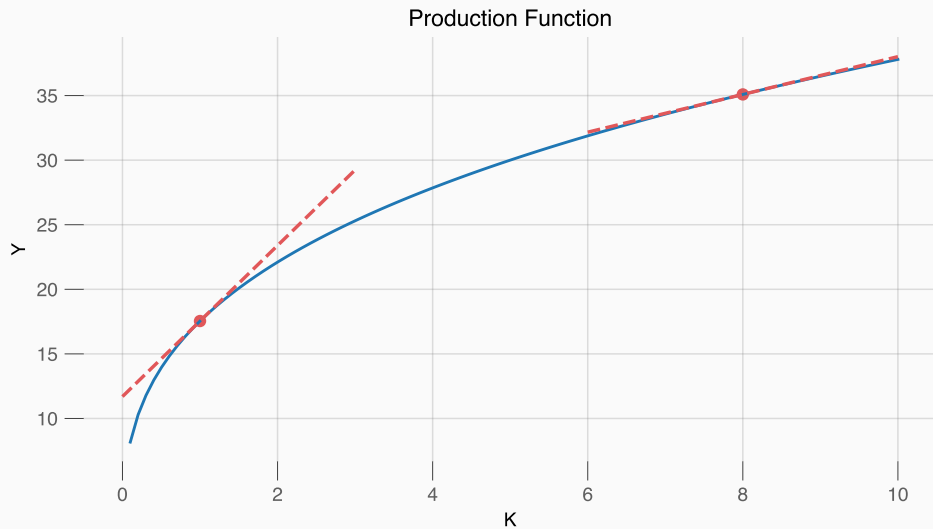
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Diminishing Returns – what happens as we keep increasing K ?



Marginal Product of Capital (MPK)

- ▶ Time for a little bit of calculus: take the partial derivative of the production function with respect to capital:

$$MPK = \frac{\partial Y}{\partial K}$$

This will tell use how much Y changes when we change K

- ▶ In our application, we need the power rule:

$$\text{Let } y = x^a, \text{ then } \frac{\partial y}{\partial x} = ax^{a-1}$$

$$\frac{\partial Y}{\partial K} = \frac{1}{3}AK^{-2/3}L^{2/3}$$

- ▶ This is also the slope of the production function we just plotted
- ▶ As we saw graphically, the slope gets flatter as K increases
 - Check it! I used $A = 6$, $L = 5$, $\alpha = \frac{1}{3}$
- ▶ We call this **diminishing returns** to capital.
- ▶ We can do a similar exercise for L and find a similar outcome

► We just constructed the first piece of the **simple production economy**

1. The Production Function ✓

2. Factor Supply

- Fixed endowment of labor: $L = \bar{L}$
- Fixed endowment of capital: $K = \bar{K}$

3. Producer behavior

- Competitive producers maximize profits
- These producers demand capital and labor

4. Equilibrium

- Set supply = demand
- Solve for prices and output

Producers

- ▶ The producer takes the price of its output (p) and the prices of inputs as given.
- ▶ **Problem:** How much labor and capital should the firm hire?
- ▶ To answer, we need to know what the marginal benefit of hiring a worker is.
- ▶ What is a firm's revenue?

$$Rev = pY = pF(K, L)$$

- ▶ We want to know what the benefit of hiring one additional worker is, so we take the derivative:

$$p \frac{\partial F(K, L)}{\partial L} = \frac{2}{3} AK^{1/3} L^{-1/3}$$

- ▶ Recall diminishing returns – first worker adds a lot, each additional worker, less

- ▶ Now we need the firm's cost:

$$Cost = wL + rK$$

where $w \equiv$ the wage and $r \equiv$ the price of capital

- ▶ We can take the derivative of this with respect to labor and get:

$$\frac{\partial Cost}{\partial L} = w$$

This is marginal cost of labor

- ▶ How do people make decisions in economics?

- Marginal cost = marginal benefit!

- ▶ So:

$$\frac{2}{3}AK^{1/3}L^{-1/3} = w$$

- ▶ Similarly:

$$\frac{1}{3}AK^{-2/3}L^{2/3} = r$$

The Profit Function

- ▶ We could do the calculus all separately, or we could do it all at once.
- ▶ We can construct a producer's profit function:

$$\Pi = pY - wL - rK$$

- ▶ Given p , r , and w , the producer chooses K and L to maximize profit:

$$\max_{\{K,L\}} pY - wL - rK$$

such that $Y = AK^{1/3}L^{2/3}$

- Take the derivatives and set them equal to zero (also called first-order conditions):

$$\frac{\partial Y}{\partial L} : \frac{2}{3}AK^{1/3}L^{-1/3} = w$$

$$\frac{\partial Y}{\partial K} : \frac{1}{3}AK^{-2/3}L^{2/3} = r$$

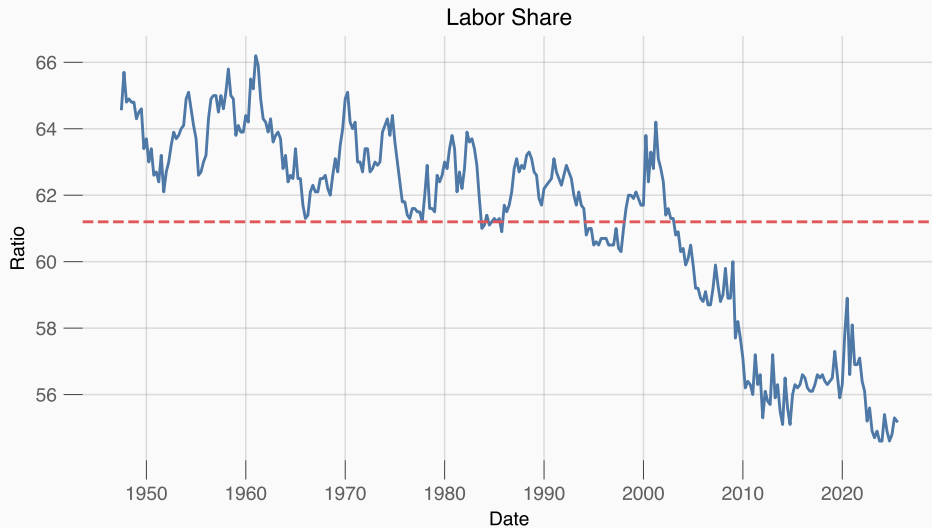
- For greater intuition, note that $AK^{1/3}L^{-1/3} = \frac{Y}{L}$, and that $AK^{-2/3}L^{2/3} = \frac{Y}{K}$.
Then:

$$\frac{wL}{pY} = \frac{2}{3}$$

$$\frac{rK}{pY} = \frac{1}{3}$$

- So the share of revenue (pY) paid to capital is $1/3$
- The share of revenue paid to labor is $2/3$

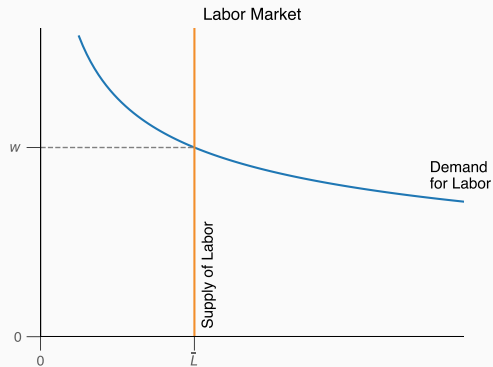
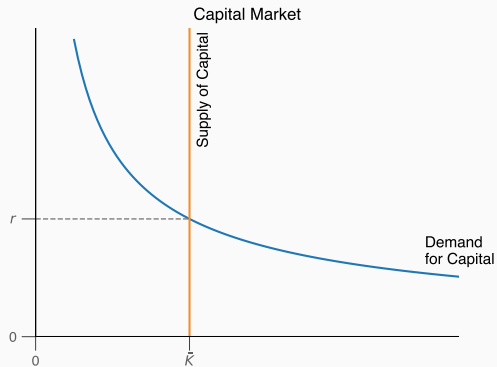
The Data – $2/3$ was a decent approximation until recently



1. The Production Function ✓
2. Factor Supply ✓
 - Fixed endowment of labor: $L = \bar{L}$
 - Fixed endowment of capital: $K = \bar{K}$
3. Producer behavior ✓
 - Competitive producers maximize profits
 - These producers demand capital and labor
4. Equilibrium
 - Set supply = demand
 - Solve for prices and output

- ▶ Producer decisions are in fact *demand* curves
 - $\downarrow w \longrightarrow \uparrow L$
 - $\downarrow r \longrightarrow \uparrow K$
- ▶ Producers have downward-sloping demand for K and L
- ▶ We fixed the supply of K and L
- ▶ We now equate supply and demand

Graphical Equilibrium



Practice Problems

Suppose that $A = 2$, $\alpha = 1/3$, $\bar{K} = 100$, $\bar{L} = 50$, and $p = 1$. Let the production function be a constant returns-to-scale Cobb Douglas function

- (1) Compute Y
- (2) Derive the marginal product of capital (MPK) and the marginal product of labor (MPL)
- (3) Find the factor prices r and w
- (4) Verify that capital expenditure relative to output is α . Verify that labor expenditure relative to output is $1 - \alpha$
- (5) Suppose A increases by 10%. By what percentage do Y , r , and w change?

(1) $Y = 2 \times 100^{1/3} \times 50^{2/3} \approx 126$

(2) The profit function is: $\Pi = pY - wL - rK$. Take the derivatives with respect to K and L:

$$\frac{\partial \Pi}{\partial K} : r = \alpha AK^{\alpha-1} L^{1-\alpha}$$

$$\frac{\partial \Pi}{\partial L} : w = (1 - \alpha) AK^{\alpha} L^{-\alpha}$$

(3) Plug-in our specific calibration:

$$r = \frac{1}{3} \times 2 \times 100^{-2/3} \times 50^{2/3} \approx 0.42$$

$$w = \frac{2}{3} \times 2 \times 100^{1/3} \times 50^{-1/3} \approx 1.68$$

(4) Plug-in our solutions:

$$\alpha = \frac{rK}{pY}$$
$$\frac{1}{3} = \frac{0.42 \times 100}{126}$$
$$\frac{1}{3} = \frac{1}{3} \checkmark$$

$$1 - \alpha = \frac{wL}{pY}$$
$$\frac{2}{3} = \frac{1.68 \times 50}{126}$$
$$\frac{2}{3} = \frac{2}{3} \checkmark$$

(5) For Y : $100 \times \frac{(1.1)A_0K^\alpha L^{1-\alpha} - A_0K^\alpha L^{1-\alpha}}{A_0K^\alpha L^{1-\alpha}} = 10\%$

For r : $100 \times \frac{\alpha(1.1)A_0K^{\alpha-1}L^{1-\alpha} - A_0K^{\alpha-1}L^{1-\alpha}}{A_0K^{\alpha-1}L^{1-\alpha}} = 10\%$

For w : $100 \times \frac{(1-\alpha)(1.1)A_0K^\alpha L^{-\alpha} - (1-\alpha)A_0K^\alpha L^{-\alpha}}{(1-\alpha)A_0K^\alpha L^{-\alpha}} = 10\%$

The Production Economy – Summary

- ▶ We have three known variables: A , \bar{K} , and \bar{L}
- ▶ We usually set $p = 1$ for convenience
- ▶ Five unknowns: K , L , r , w , Y
- ▶ Five equations:
 1. Production: $Y = AK^{1/3}L^{2/3}$
 2. Capital demand: $rK = \frac{1}{3}Y$
 3. Labor demand: $wL = \frac{2}{3}Y$
 4. Capital supply: $K = \bar{K}$
 5. Capital demand: $L = \bar{L}$
- ▶ Next week: read chapters 9.3-9.4